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MODELING THE MIMIMUM TIME NEEDED TO ECONOMIC MATURITY

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Abstract

A general equilibrium model has been constructed in a stochastic endogenous growth economy with the capital-labor ratio driven by an Itô-Lévy diffusion process. In particular, formal definition of the minimum-time needed to economic maturity is identified in the model and closed-form solutions of optimal savings strategy and optimal tax rates are established when the representative agent is assumed to exhibit log preference and there is a benevolent government in the economy. Moreover, the minimum-time needed to economic maturity is explicitly derived in the sense of sub-game perfect Nash equilibrium and one can further proceed to comparative static analysis with respect to the relevant parameters of the underlying economy such as the subject discount factor, the initial level of capital stock per capita, the utility-optimal and sustainable terminal path level of capital stock per capita, the natural growth rate of population, and the exogenous level of government spending. Finally, it is worth emphasizing that we focus on underdeveloped economies such as China and the present exploration presents a baseline mathematical model for studying the optimal policies needed to reach economic maturity as soon as possible.

Keywords: Endogenous growth, Minimum-time objective, Ratchet effect, Economic maturity, Economic development

JEL classification: O11, O21, O41

1. Introduction

For any underdeveloped economy, like China, both the government and the people are motivated to choose appropriate fiscal policies and optimal investment strategies, respectively, to make the economy reach its maturity level[1] as quickly as possible. The state of "economic maturity" can be, in the category of macroeconomics, translated into the well-known von Neumann equilibrium (see, Neumann, 1945-1946; Yano, 1998), "turnpike"[2] (e.g., Hicks, 1961; Radner, 1961; McKenzie, 1963;

[1] Undoubtedly, it should reflect not only high speed of economic growth but also high quality of economic development. More about this topic of growth and development, one can refer to Solow (2003).
[2] Related references see, Joshi (1997) and Dai (2012) and some references therein.

Atsumi, 1965; Cass, 1966; and Gale, 1967), and the Golden Age or modified Golden Age (e.g., Phelps, 1961; Samuelson, 1965). And in turn, provided the existence of the von Neumann path or the "turnpike" of the economy, the problem facing us, including the government and the representative agent, is to choose appropriate fiscal policies and savings strategy, respectively, to effectively support the convergence of the underlying economic system, thereby implying the economy will spend almost all time staying at least in the neighborhood of the von Neumann equilibrium or the "turnpike" (see, Cass, 1966; Yano, 1984; McKenzie (1998) and references therein), which indeed represents the maximal and sustainable terminal path level (e.g., Kurz, 1965; Dai, 2012) of the corresponding economy in the present model.

The current paper is not devoted to confirm the existence of the unique von Neumann path or the well-known "turnpike" of an aggregate endogenous growth economy equipped with AK production technology (e.g., Barro, 1990; Rebelo, 1991; Turnovsky, 2000; Wälde, 2011). Indeed, the major goal of this paper is to *explicitly compute* the minimum time needed to reach the "economic maturity" for an underdeveloped economy and in an uncertain environment. Moreover, it's easy to notice that our paper is a natural extension of the seminal and interesting paper of Kurz (1965)[3], where optimal paths of capital accumulation under the minimum time objective are thoroughly investigated. It is, nevertheless, worth emphasizing that our results are based upon the general equilibrium framework and the minimum time is endogenously determined provided the welfare of the representative agent is maximized[4].

The advantage of the method used here is that the endogenous time or the minimum time needed to "economic maturity" can be *explicitly computed*[5] in some conditions, e.g., when the preference or the criterion of *the modified Radner* (1961) *fashion* is employed. Noting that the minimum time is endogenously determined by the optimal savings strategy of the representative agent and the optimal taxation policies of the government, thereby implying that the endogenous time can be completely characterized, which obviously makes our understanding of the minimum time needed to reach "economic maturity" for an underdeveloped economy much easier.

The current paper proceeds as follows. Section 2 introduces the general model and the basic idea behind the macroeconomic model. Section 3 computes the endogenous time in preference manifold of the modified Radner fashion. There is a brief concluding section. All proofs, unless otherwise noted in the text, appear in the Appendix.

[3] It is regarded as a continuation of Srinivasan's work (1962) in a certain sense.
[4] In other words, high speed of economic growth is based upon the high quality of economic development.
[5] That is, a simple formula is supplied for the first time. And also, it is easy to see that the maximal terminal path level of capital stock per capita is utility-optimal and simultaneously determined with the endogenous time in the present model.

2. The general model

2.1. Modified Radner preference

In order to determine the minimum time needed to reach the so-called von Neumann path or "economic maturity" for an underdeveloped economy, the following criterion is naturally investigated.

This criterion has been widely employed to prove the well-known turnpike theorems, and noting that it is pioneered by Radner (1961), we call it *the Radner fashion*. However, it is worth noting that the discount factor is naturally incorporated into the criterion while it is excluded in the seminal paper of Radner, that is, we employ *the modified Radner fashion* in the current paper. Formally, given a probability space (Ω, F, P), the corresponding problem can be written as, $\sup_{\tau \in \mathrm{T}} E\left[e^{-\rho\tau} u\left(c(\tau)\right) \mathbf{1}_{\{\tau<\infty\}}\right]$, where T denotes admissible stopping times, $0<\rho<1$ denotes subjective discount factor, c denotes consumption per capita, $u: R_{+} \to R$ is a strictly concave instantaneous utility function, and $\mathbf{1}_{\{\tau<\infty\}}$ represents the indicator function of set $\{\omega \in \Omega; \tau(\omega) < \infty\}$.

REMARK 2.1. It is easy to see from our specification that there is a natural one-to-one correspondence between the optimal stopping time and the minimum time needed to "economic maturity" for any underdeveloped economy. Notice that *the modified Radner fashion* reflects some psychological effects[6], so it also would be called the "peak preference"[7].

2.2. Computation algorithm of the endogenous time

There are alternative goals for the government, that is, government is either motivated to choose taxation policies so as to maximize the welfare of the representative agent or directly minimize the time needed to "economic maturity". The order of action is like this: the government moves first to choose optimal taxation policies, then the representative agent determines the minimum time needed to "economic maturity" based upon the optimal taxation policies, and finally, the representative agent chooses optimal savings strategy conditional on the optimal taxation policies and the endogenous time horizon representing the process leading to "economic maturity".

[6] For example, one can refer to the well-known "the Ratchet effect" in traditional consumption theory.

[7] That is, the representative agent pursues the highest level of utility or welfare of any single period. And it is just the highest level of the welfare that represents the corresponding state of "economic maturity" in the current model.

Therefore, based on the backward induction rationality principle, we introduce the following computation algorithm of the current model:

STEP 1: *The representative agent chooses optimal savings strategy given the taxation policies and the finite time horizon of the program.*
STEP 2: *Based on the results of Step* 1*, the representative agent will determine the minimum time needed to reach "economic maturity".*
STEP 3a: *If the goal of the government is to maximize the welfare of the representative agent, thus based upon the results of Steps* 1 *and* 2*, the optimal tax rates are derived.*
STEP 3b: *If the goal of the government is to directly minimize the time needed to "economic maturity" derived in Step* 2*, then the corresponding optimal taxation policies are established.*
STEP 4: *This step is necessary only when Step* 3a *is chosen. Substituting the optimal tax rates into the endogenous time derived in Step* 2 *so that the minimum time needed to "economic maturity" is finally and completely derived.*

3. Minimum-time needed to economic maturity

3.1. Firm

In the current paper, we introduce the following Cobb-Douglas type production function,

$$Y(t) = G_p(t)^{\alpha} K(t)^{1-\alpha}, \ 0 < \alpha < 1 \tag{1}$$

where K denotes the aggregate capital stock and G_p represents the flow of services from government spending on the economy's infrastructure. Particularly, suppose that these services are not subject to congestion so that G_p is a pure public good. Further to put $G_p = g_p Y$[8], that is government will claim a fraction, g_p, of aggregate output Y, for expenditure on infrastructure. And, in particular, to make things easier and without loss of any generality, g_p will be assumed to be exogenously given with $0 < g_p < 1$ throughout the paper, then the production function in (1) can be rewritten as,

$$Y(t) = g_p{}^{\alpha/1-\alpha} K(t), \text{ or } y(t) = g_p{}^{\alpha/1-\alpha} k(t) \tag{1'}$$

which reveals that the Cobb-Douglas type function given in (1) rather exhibits AK production technology, which indeed assures ongoing economic growth. Therefore, equilibrium wage rate is equal to zero and equilibrium return to capital reads as follows,

[8] This specification follows from Turnovsky (2000).

$$r_k = g_p^{\alpha/1-\alpha}, \tag{2}$$

where the depreciation rate is assumed to be zero for the sake of simplicity.

3.2. Representative agent

It is assumed that the economy consists of $L(t)$ identical individuals at time t, each of whom possesses perfect foresight. Suppose that $\{B(t)\}_{0\le t\le T}$ is a standard Brownian motion defined on the following filtered probability space $(\Omega^{(B)}, F^{(B)}, \{F_t^{(B)}\}_{0\le t\le T}, P^{(B)})$ with $\{F_t^{(B)}\}_{0\le t\le T}$ the $P^{(B)}-$augmented filtration generated by $\{B(t)\}_{0\le t\le T}$ with $F^{(B)} = F_T^{(B)}$. Furthermore, we assume that a Poisson random measure $N(dt,dz)$ associated with a Lévy process is defined on the stochastic basis $(\Omega^{(N)}, F^{(N)}, \{F_t^{(N)}\}_{0\le t\le T}, P^{(N)})$. And we denote by $N(dt,dz) = N(dt,dz) - n(dz)dt$ the compensated Poisson random measure associated with a Lévy process $\eta(t) \equiv \int_0^t \int_{R_0} zN(ds,dz)$[9] with jump measure $N(dt,dz)$ and Lévy measure $n(O) = E[N([0,1],O)]$ for $O \in \mathrm{B}(R_0)$, i.e., O is a Borel set with its closure $\overline{O} \subset R_0$, where $R_0 \equiv R - \{0\}$. In what follows, our reference stochastic basis will be $(\Omega, F, \{F_t\}_{0\le t\le T}, P)$ with $\Omega = \Omega^{(B)} \times \Omega^{(N)}$, $F = F^{(B)} \otimes F^{(N)}$, $F_t = F_t^{(B)} \otimes F_t^{(N)}$ and $P = P^{(B)} \otimes P^{(N)}$ and also the underlying probability measure space is assumed to satisfy the so-called "usual conditions"[10]. Based on the above constructions and assumptions, we now define[11],

$$dL(t) = L(t^-)\left[ndt + \sigma dB(t) + \int_{R_0} \gamma z\overline{N}(dt,dz)\right], \tag{3}$$

where n denotes the natural growth rate of population, $\sigma \in R_0$ is an exogenously given constant, $\gamma z > -1$ a.s.$-\nu$, $B(0) = 0$ a.s.$-P$ and[12],

[9] This is a pure jump process and it is indeed a martingale and a strong Markov process. This process will capture some characteristic of reality that cannot be captured by standard Brownian motions.
[10] That is, the probability space is complete and the filtration satisfies right continuity.
[11] This is a natural extension of the specification of Merton (1975). And here we apply Lévy diffusion (usually, a Lévy process can be written as a linear combination of time, Brownian motion and a pure jump process as shown in (3)) to macroeconomics, which can be regarded as reasonable via noting the properties of both Lévy diffusions and macroeconomic phenomenon. For example, if we consider the population process *L(t)* as labor supply, then there will be some jumps of labor supply during the period of economic crisis and the specification of (3) reasonably captures this effect in reality.
[12] About this definition, one can refer to the standard textbook of Øksendal and Sulem (2005). The definition in (4) shows us different choices of jump *z* corresponding to different random measures.

$$\overline{N}(dt,dz) \equiv \begin{cases} N(dt,dz) - \nu(dz)dt \equiv N(dt,dz), & |z| < Z \\ N(dt,dz), & |z| \geq Z \end{cases} \tag{4}$$

for some $Z \in [0,\infty]$. As usual, we define the following law of motion of capital accumulation,

$$\dot{K}(t) = \mu[(1-\tau_k)r_k K(t) - (1+\tau_c)C(t)] = \mu g_p^{\alpha/1-\alpha}[(1-\tau_k) - (1+\tau_c)(1-g_p-r_s)]K(t), \tag{5}$$

where $\mu \in R_0$ is some exogenously given parameter, r_k denotes the equilibrium return to capital given in (2), τ_k denotes tax rate on capital income, t_c represents consumption tax rate, C denotes aggregate consumption level and r_s denotes the savings rate. Hence, combining (3) with (5) and by applying Itô formula for Itô-Lévy process, we get,

$$\begin{aligned} dk(t) = &\left\{\mu g_p^{\alpha/1-\alpha}[(1-\tau_k) - (1+\tau_c)(1-g_p-r_s)] - n + \sigma^2\right\}k(t^-)dt \\ &+ \int_{|z|<Z} \tfrac{(\gamma z)^2}{1+\gamma z}\nu(dz)k(t^-)dt - \sigma k(t^-)dB(t) - k(t^-)\int_{R_0} \tfrac{\gamma z}{1+\gamma z}\overline{N}(dt,dz), \end{aligned} \tag{6}$$

Without loss of any generality, we put $Z = \infty$, then by (4), (6) becomes[13],

$$\begin{aligned} dk(t) = &\left\{\mu g_p^{\alpha/1-\alpha}[(1-\tau_k) - (1+\tau_c)(1-g_p-r_s)] - n + \sigma^2 + b\right\}k(t^-)dt \\ &- \sigma k(t^-)dB(t) - k(t^-)\int_{R_0} \tfrac{\gamma z}{1+\gamma z}N(dt,dz), \end{aligned} \tag{7}$$

where[14],

$$b \equiv \int_R \tfrac{(\gamma z)^2}{1+\gamma z}\nu(dz), \tag{8}$$

Suppose that the representative agent performs log preference and the intertemporal objective function is specifically given as[15],

[13] Noting that we usually consider *finite jump* when modeling uncertainty in reality by Lévy processes, we thus employ compensated Poisson random measure based on the definition given by (4). Hence, we can rewrite (6) as (7). The formula in (7) shows that the law of motion of capital accumulation will follow a Lévy process as long as the stochastic growth of population or labor supply is driven by a Lévy process. That is to say, the uncertainty of capital accumulation not only results from the continuous fluctuation driven by Brownian motion but also results from discontinuous jump driven by a pure jump process. For example, one may model technology fluctuation by the standard Brownian motion and model technology breakthrough by the pure jump process. Finally, similar specification has been employed by Dai et al. (2013).

[14] The formula in (8) is the Lévy integral of a particular jump and one can regard it as an exogenous constant in the present model. And we denote it by *b* just for notational simplicity. Indeed, the constant in (8) just results from the employment of Itô formula for Itô-Lévy process, which implies that it does not imply any meaningful economic intuition in the present model.

[15] As you can see, the utility or welfare of the representative agent just relies on individual consumption in the present environment and we denote the corresponding expected discounted utility by (9). The formula in (9) indeed reflects an

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$$U = E\left[\int_0^{\hat{\tau}} e^{-\rho(s+t)} \ln c(t)dt + U^{\hat{\tau}}\right] = E\left[\int_0^{\hat{\tau}} e^{-\rho(s+t)} \ln\left((1- g_p - r_s)g_p{}^{\alpha/1-\alpha}k(t)\right)dt + U^{\hat{\tau}}\right]. \quad (9)$$

where E denotes expectation operator with respect to probability measure P, ρ is the subjective discount factor, $\forall 0 \le s < \hat{\tau}$ and $\hat{\tau}$ is an $F-$optimal stopping time, which with the term $U^{\hat{\tau}}$ are simultaneously determined by the following optimal stopping problem of *the modified Radner fashion*,[16]

$$\hat{g}(\tau, k(\tau)) \equiv \sup_{\tau \in \mathrm{T}} E^{(s,k)}\left[e^{-\rho(s+\tau)} \ln\left(g_p{}^{\alpha/1-\alpha}k(\tau)\right)\mathbf{1}_{\{\tau<\infty\}}\right] = E^{(s,k)}\left[e^{-\rho(s+\hat{\tau})} \ln\left(g_p{}^{\alpha/1-\alpha}k(\hat{\tau})\right)\mathbf{1}_{\{\hat{\tau}<\infty\}}\right]. \quad (10)$$

subject to (7), $E^{(s,k)}$ denotes expectation operator, and $\mathrm{T} \equiv \{F - \text{stopping times}\}$.

Now it follows from Step 1 introduced in section 2.2 that we are to consider the following stochastic optimal control problem (see, Dai, 2012) facing the representative agent,

$$\max_{0<r_s<1} E\left[\int_0^{\hat{\tau}} e^{-\rho(s+t)} \ln\left((1- g_p - r_s)g_p{}^{\alpha/1-\alpha}k(t)\right)dt + U^{\hat{\tau}}\right]. \quad (9')$$

subject to (7) with $\hat{\tau}$ and $U^{\hat{\tau}}$ exogenously given. In other words, we now determine the optimal savings strategy of the representative agent provided the horizon of the planning and the terminal utility of the representative agent owing to the computation algorithm presented in Section 2. And it will be established in the sense of sub-game perfect Nash equilibrium. We prove that there exists a

important specification of the present model. First, the horizon of the economy is stochastically endogenously determined and one may notice certain similarity of the present approach to those literatures studying endogenous lifetime or endogenous longevity in growth models (see, Chakraborty, 2004; de la Croix and Ponthiere, 2010, and among others); second, in the classical neoclassical aggregate growth models, national income is just equal to the wealth of the representative household and here we represents individual consumption by the product of consumption rate (noting that the national income in the present economy is distributed to individual consumption, individual savings and also government spending) and individual wealth (or aggregate output); third, the terminal utility of the representative agent is also endogenously determined in the current model and it proves to be quite helpful in the following computation of the equilibrium minimum-time needed to economic maturity. To summarize, both the horizon of the planning problem and the terminal utility are endogenously determined in (9), which essentially distinguishes itself from the usual specification of objective function in existing literatures.

[16] It is especially worth emphasizing that we assume that there is no savings in the terminal period of the planning and hence the terminal utility of the representative agent totally comes from the national income or total output (or individual wealth in the present aggregate growth model), i.e., output is purely consumed in the terminal period. We argue in (9) that the terminal utility of the representative agent is endogenously determined and the formula in (10) shows that the terminal utility is actually determined by an optimal stopping problem. In other words, the specification in (10) shows that the horizon of the planning and the optimal terminal utility can be simultaneously determined by employing optimal stopping theory widely used in mathematical finance. To sum up, the optimal stopping problem defined in (10) shows the advantage of the technique widely used in mathematical finance because we can simultaneously determine the endogenous lifetime of the representative agent or the horizon of the underlying planning and the terminal optimal utility of the representative agent that captures the well-known Ratchet effect in traditional consumption theory. And it would be easily noticed that (10) is indeed a specific realization of the general model in Section 2.1.

continuously differential function $V(t,k(t))$, satisfying the following stochastic Bellman partial differential equation (PDE)[17],

$$\begin{aligned}&-V_t(t,k(t))-\tfrac{1}{2}\sigma^2k^2(t)V_{kk}(t,k(t))\\&-\int_{R_0}\left[V\left(t,k(t)-\tfrac{\gamma z}{1+\gamma z}k(t)\right)-V(t,k(t))+\tfrac{\gamma z}{1+\gamma z}k(t)V_k(t,k(t))\right]n(dz)\\&=\max_{0<r_s<1}\left\{\begin{matrix}\exp(-\rho(s+t))\ln[(1-g_p-r_s)g_p{}^{\alpha/1-\alpha}k(t)]+V_k(t,k(t))k(t)\\ \times\left\{\mu g_p{}^{\alpha/1-\alpha}[(1-\tau_k)-(1+\tau_c)(1-g_p-r_s)]-n+\sigma^2+b\right\}\end{matrix}\right\}.\end{aligned} \tag{11}$$

with the boundary condition,

$$V(\hat{\tau},k(\hat{\tau}))=U^{\hat{\tau}}, \tag{12}$$

Thus, we get,

LEMMA 1. *Conditional on the above constructions and assumptions, and up to the present step, we obtain the optimal savings rate as* $\hat{r}_s=1-g_p-\frac{\rho}{\mu g_p{}^{\alpha/1-\alpha}(1+\tau_c)}$. *Moreover, the value function* $V(t,k(t))$ *satisfies the following boundary condition* $V(\hat{\tau},k(\hat{\tau}))=\exp(-\rho(s+\hat{\tau}))[C_1+\rho^{-1}\ln k(\hat{\tau})]=U^{\hat{\tau}}$ *with,*

$$C_1\equiv\rho^{-1}\left\{\ln\tfrac{\rho}{\mu(1+\tau_c)}+\rho^{-1}[\mu g_p{}^{\alpha/1-\alpha}(1-\tau_k)-n+\sigma^2+b]-1-\tfrac{1}{2}\sigma^2\rho^{-1}+\rho^{-1}\int_{R_0}\left(\ln\tfrac{1}{1+\gamma z}+\tfrac{\gamma z}{1+\gamma z}\right)n(dz)\right\}.$$

Proof. See Appendix A. ■

REMARK 3.1. The boundary condition shown in Lemma 1 will be very useful in computing the closed-form solution of the endogenous time as is shown in the sequel.

Now, by applying Step 2 of the computation algorithm, we can calculate the term $U^{\hat{\tau}}$ and the optimal stopping time $\hat{\tau}$ given in (9). Firstly, via applying Lemma 1, (7) can be rewritten as,

$$\begin{aligned}dk(t)=&\left\{\mu g_p{}^{\alpha/1-\alpha}[(1-\tau_k)-(1+\tau_c)(1-g_p-\hat{r}_s)]-n+\sigma^2+b\right\}k(t^-)dt\\&-\sigma k(t^-)dB(t)-k(t^-)\int_{R_0}\tfrac{\gamma z}{1+\gamma z}N(dt,dz),\end{aligned} \tag{7'}$$

Let $Y(t)\equiv(s+t,k(t))^T$, $Y(0)\equiv(s,k)^T$, then the generator of $Y(t)$ reads as follows,

$$\begin{aligned}A\phi(s,k)=&\tfrac{\partial\phi}{\partial s}+\left\{\mu g_p{}^{\alpha/1-\alpha}[(1-\tau_k)-(1+\tau_c)(1-g_p-\hat{r}_s)]-n+\sigma^2+b\right\}k\tfrac{\partial\phi}{\partial k}\\&+\tfrac{1}{2}\sigma^2k^2\tfrac{\partial^2\phi}{\partial k^2}+\int_{R_0}[\phi(s,k-\tfrac{\gamma z}{1+\gamma z}k)-\phi(s,k)+\tfrac{\gamma z}{1+\gamma z}k\tfrac{\partial\phi}{\partial k}]n(dz),\end{aligned} \tag{13}$$

[17] This (continuous-time) Bellman equation is a natural extension of that corresponding to standard Brownian motions. Consequently, Lévy integral is also included in the formula in (11) owing to the Lévy process denoted by (7). And one can also refer to Dai et al. (2013) for similar specification.

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for $\forall \phi \in C^2(R^2)$. If we try a function ϕ of the form, $\phi(s,k) = e^{-\rho s}k^{\beta}$, for some constant $\beta \in R$. We obtain, $A\phi(s,k) = e^{-\rho s}k^{\beta}\left(-\rho + \beta\left\{\mu g_p{}^{\alpha/1-\alpha}[(1-\ \tau_k)-(1+\ \tau_c)(1-\ g_p-\ \hat{r}_s)]-n+\sigma^2+b\right\}\right)$
$+\frac{1}{2}e^{-\rho s}k^{\beta}\sigma^2\beta(\beta-1)+e^{-\rho s}k^{\beta}\int_{R_0}[(\frac{1}{1+\gamma z})^{\beta}-1+\frac{\gamma z\beta}{1+\gamma z}]\nu(dz) = e^{-\rho s}k^{\beta}h(\beta)$,
in which,

$$h(\beta) \equiv -\rho + \beta\left\{\mu g_p{}^{\alpha/1-\alpha}[(1-\ \tau_k)-(1+\ \tau_c)(1-\ g_p-\ \hat{r}_s)]-n+\sigma^2+b\right\}$$
$$+\frac{1}{2}\sigma^2\beta(\beta-1)+\int_{R_0}[(\frac{1}{1+\gamma z})^{\beta}-1+\frac{\gamma z\beta}{1+\gamma z}]\nu(dz). \tag{14}$$

Notice that $h(0) = -\rho < 0$ and $\lim_{|\beta|\to\infty} h(\beta) = \infty$. Therefore, there exists $\beta > 0$ such that $h(\beta) = 0$ and with this value of β, we put[18]

$$\phi(s,k) = \begin{cases} e^{-\rho s}Ck^{\beta}, & (s,k) \in D \\ e^{-\rho s}\ln(g_p{}^{\alpha/1-\alpha}k), & (s,k) \notin D \end{cases} \tag{15}$$

F
or some constant $C > 0$ and the continuation region D, to be determined. Thus, if we define $g(s,k) \equiv e^{-\rho s}\ln(g_p{}^{\alpha/1-\alpha}k)$. Then we have, by (13),

$$Ag(s,k) = e^{-\rho s}\left\{-\rho\ln(g_p{}^{\alpha/1-\alpha}k)+\mu g_p{}^{\alpha/1-\alpha}[(1-\ \tau_k)-(1+\ \tau_c)(1-\ g_p-\ \hat{r}_s)]\right.$$
$$\left.-n+\frac{1}{2}\sigma^2+b+d\right\} > 0 \tag{16}$$

$$\Leftrightarrow k < g_p{}^{-\alpha/1-\alpha}\exp\left\{\frac{\mu g_p{}^{\alpha/1-\alpha}[(1-\ \tau_k)-(1+\ \tau_c)(1-\ g_p-\ \hat{r}_s)]-n+\frac{1}{2}\sigma^2+b+d}{\rho}\right\}$$

Where

$$d \equiv \int_{R_0}\left(\ln\frac{1}{1+\gamma z}+\frac{\gamma z}{1+\gamma z}\right)n(dz), \tag{17}$$

Hence, we can define,

[18] It is especially worthwhile mentioning that we are prepared to formulate a set of sufficient conditions so that a given function actually coincides with the value function or the optimal utility of the representative agent and that a corresponding stopping time actually is optimal. And we need employ the well-known Integrovariational Inequalities for Optimal Stopping (see, Øksendal and Sulem, 2005). As shown in (15), we will first find out the continuation region, which is indeed a set of initial conditions where the function we will construct satisfy the necessity of optimization, and then we establish the corresponding function on this continuation region such that this function is actually optimal, that is, it coincides with the value function (or the optimal utility of the representative agent). To sum up, the construction shown in (15) shows the underlying purpose of finding out the appropriate continuation region of initial conditions on which the desired function is identified to satisfy optimality.

$$U=\left\{(s,k);k<g_p^{-\alpha/1-\alpha}\exp\left(\frac{\mu g_p^{\alpha/1-\alpha}[(1-\ \tau_k)-(1+\tau_c)(1-\ g_p-\ \hat{r}_s)]-n+\frac{1}{2}\sigma^2+b+d}{\rho}\right)\right\}, \tag{18}$$

Thus, it is natural to guess that the continuation region D has the form,

$$D=\{(s,k);0<k<\hat{k}\}. \tag{19}$$

for some $\hat{k}$ such that $U\subseteq D$, i.e.,

$$\hat{k}\geq g_p^{-\alpha/1-\alpha}\exp\left(\frac{\mu g_p^{\alpha/1-\alpha}[(1-\ \tau_k)-(1+\tau_c)(1-\ g_p-\ \hat{r}_s)]-n+\frac{1}{2}\sigma^2+b+d}{\rho}\right), \tag{20}$$

Thus, (15) can be rewritten as follows[19],

$$\phi(s,k)=\begin{cases}e^{-\rho s}Ck^{\beta}, & 0<k<\hat{k}\\ e^{-\rho s}\ln(g_p^{\alpha/1-\alpha}k), & k\geq\hat{k}\end{cases} \tag{21}$$

where $\hat{k}>0$ and C remain to be determined. Moreover, continuity and differentiability of ϕ at $k=\hat{k}$ implies that $C(\hat{k})^{\beta}=\ln(g_p^{\alpha/1-\alpha}\hat{k})$ and $C\beta(\hat{k})^{\beta-1}=(\hat{k})^{-1}$, which hence reveal that,

$$\frac{C(\hat{k})^{\beta}}{C\beta(\hat{k})^{\beta}}=\ln(g_p^{\alpha/1-\alpha}\hat{k})\Leftrightarrow\hat{k}=g_p^{-\alpha/1-\alpha}\exp(\tfrac{1}{\beta}) \tag{22}$$

$$C=\tfrac{1}{\beta}(\hat{k})^{-\beta}=\tfrac{1}{\beta}[g_p^{-\alpha/1-\alpha}\exp(\tfrac{1}{\beta})]^{-\beta}. \tag{23}$$

To summarize, we have,
LEMMA 2. *Under the above assumptions and constructions, if* $\sigma<0$, $-1<\gamma z<0$ *a.s.*$-\nu$, $\int_{R_0}[(\frac{1}{1+\gamma z})^{\beta}-1]^2\nu(dz)<\infty$, *and also* $n-\frac{1}{2}\sigma^2-\int_R\gamma z\nu(dz)<\mu g_p^{\alpha/1-\alpha}[(1-\ \tau_k)-(1+\tau_c)(1-\ g_p-\ \hat{r}_s)]$ $\leq\min\left\{\rho+n-\sigma^2-b,\rho+n-\frac{3}{2}\sigma^2-\frac{1}{2}\int_{R_0}[(\frac{1}{1+\gamma z})^2-1]\nu(dz)-\int_R\gamma z\nu(dz)\right\}$ *with,*

$$\Big|2\beta\mu g_p^{\alpha/1-\alpha}[(1-\ \tau_k)-(1+\tau_c)(1-\ g_p-\ \hat{r}_s)]+2\beta\int_R\gamma z\nu(dz)$$
$$-2\beta n+(\beta+2\beta^2)\sigma^2+\int_{R_0}[(\tfrac{1}{1+\gamma z})^{2\beta}-1]\nu(dz)\Big|<\infty,$$

[19] Combining (15) with (19) leads us to the formula in (21). That is, one may regard (21) as a special realization of that in (15).

where b *is defined in* (8). *Then we obtain the optimal* F_t *– stopping time,* $\hat{\tau} \equiv \inf\{t \geq 0; k(t) = \hat{k}\}$. *In other words,* $\hat{g}(s,k) = e^{-\rho s} \frac{1}{\beta}(\hat{k})^{-\beta} k^{\beta} = U^{\hat{\tau}}$, *which is a supermeanvalued majorant of* $g(s,k)$ *with* $\hat{k}$ *given by* (22) *and* β *is a solution of* $h(\beta) = 0$ *in* (14).
Proof. See Appendix B. ■

REMARK 3.2. Obviously, $\hat{k}$ given in (22) would be seen as the maximal and sustainable terminal path level of capital stock per capita that is optimal in the sense of the modified Radner preference. Noting that $\hat{k}$ is endogenously determined in the current paper while the maximal terminal path level usually exogenously specified in existing literatures, for instance, the interesting paper of Kurz (1965). Consequently, we argue that the advantage of the optimal stopping theory employed here is that it is available to make the minimum time needed to "economic maturity" and the utility-optimal and sustainable terminal path level of capital stock per capita simultaneously and endogenously determined.

3.3. Government

In the absence of debt, tax revenues and government expenditures must satisfy the following balanced budget constraint,

$$\tau_k r_k K(t) + \tau_c c(t) L(t) = g_p Y(t), \tag{24}$$

Using (1'), (2) and Lemma 1, (24) can be rewritten as,

$$\tau_k + \tau_c (1 - g_p - \hat{r}_s) = g_p. \tag{25}$$

Now, following from Step 3a shown in section 2.2, we consider the following case,
CASE 1. *The goal of the government is to maximize the welfare of the representative agent.*
Substituting (25) into (7') gives[20],

$$dk(t) = k(t^-)\left[(\mu g_p^{\alpha/1-\alpha} \hat{r}_s - n + \sigma^2 + b)dt - \sigma dB(t) - \int_{R_0} \tfrac{\gamma z}{1+\gamma z} N(dt,dz) \right], \tag{26}$$

And hence the stochastic optimal control problem facing the government can be expressed as follows[21],

[20] We specifically consider a benevolent government that can only raise funds by levying distortionary taxes, including capital-income tax and also consumption tax. Provided the irrelevance of government debt when taxes are distortionary (see, Bassetto and Kocherlakota, 2004), there is no government debt in the model and one can identify the process of capital-labor ratio denoted by (26) as the government debt in certain sense.
[21] As usual, we consider a benevolent government in the present model and the government is assumed to maximize the welfare of the representative agent except for the terminal utility of the representative agent. This is of course without

$$\max_{\substack{0<\tau_c<1\\0<\tau_k<1}} E\left[\int_0^{\hat{\tau}} e^{-\rho(s+t)} \ln\left((1- \ g_p - \ \hat{r}_s) g_p^{\ \alpha/1-\alpha} k(t)\right) dt\right]. \tag{9''}$$

subject to (26). Accordingly, the corresponding stochastic Bellman partial differential equation (PDE) amounts to[22],

$$\begin{aligned}
&-W_t(t,k(t)) - \tfrac{1}{2}\sigma^2 k^2(t) W_{kk}(t,k(t)) \\
&-\int_{R_0}\left[W\left(t,k(t)-\tfrac{\gamma z}{1+\gamma z}k(t)\right) - W(t,k(t)) + \tfrac{\gamma z}{1+\gamma z}k(t)W_k(t,k(t))\right]n(dz) \\
&= \max_{\substack{0<\tau_c<1\\0<\tau_k<1}}\left\{\exp(-\rho(s+t))\ln[(1- \ g_p - \ \hat{r}_s) g_p^{\ \alpha/1-\alpha}k(t)] + W_k(t,k(t))k(t)(\mu g_p^{\ \alpha/1-\alpha}\hat{r}_s - n + \sigma^2 + b)\right\}.
\end{aligned} \tag{27}$$

where $W(t,k(t))$ denotes the value function.

LEMMA 3. *Provided the balanced budget constraint given in* (25) *and the optimal control problem expressed in* (9”)*, then the optimal capital income tax rate is equal to* $\tau_k^* = g_p$ *and optimal consumption tax rate is zero.*

Proof. It is easily seen that the proof is quite similar to that of Lemma 1, thus we take it omitted. ■
REMARK 3.3. It is worthwhile mentioning that Lemma 3 provides us with a case against the well-known argument that capital income should not be taxed (Chamley, 1986; Judd, 2002).

Hence, by combining Lemma 3 with Lemma 1, we have,

$$\hat{r}_s = 1 - g_p - \tfrac{\rho}{\mu g_p^{\ \alpha/1-\alpha}}, \tag{28}$$

And substituting (28) and the results in Lemma 3 into (14) produce,

$$\begin{aligned}
h(\beta) &\equiv -\rho + \beta[\mu g_p^{\ \alpha/1-\alpha}(1- \ g_p) - \rho - n + \sigma^2 + b] \\
&+ \tfrac{1}{2}\sigma^2\beta(\beta-1) + \int_{R_0}[(\tfrac{1}{1+\gamma z})^\beta - 1 + \tfrac{\gamma z\beta}{1+\gamma z}]\nu(dz),
\end{aligned} \tag{14'}$$

Now, by Lemma 2, we have,

$$U^{\hat{\tau}} = e^{-\rho s}\tfrac{1}{\beta}(\hat{k})^{-\beta}k^\beta, \tag{29}$$

great loss of generality because the terminal optimal utility does not depend on the taxation policies almost everywhere in the sense of the canonical Lebesgue measure. Therefore, the problem in (9”) can be seen as a classical finite-horizon planning problem facing the benevolent government widely discussed in existing literatures.

[22] The Bellman equation denoted by (27) is based upon the diffusion process of capital accumulation in (26) and the objective function given by (9”). And it is quite similar to that of (11) and also such kind of Bellman equation appears in Dai et al. (2013). In particular, Lévy integral is included in the equation to capture the jump terms in the stochastic process of capital accumulation.

where $k = k(0) > 0$, $\hat{k}$ is given in (22), and β is a solution of equation $h(\beta) = 0$ in (14'). Combining (29) with Lemma 1 shows that, $V(\hat{\tau}, k(\hat{\tau})) = \exp(-\rho(s+\hat{\tau}))[C_1 + \rho^{-1}\ln k(\hat{\tau})] =$ $\exp(-\rho(s+\hat{\tau}))(C_1 + \rho^{-1}\ln \hat{k}) = \exp(-\rho s)\frac{1}{\beta}(\hat{k})^{-\beta}k^{\beta} = U^{\hat{\tau}}$. This implies that,

$$\hat{\tau} = \rho^{-1}\ln[\beta(\tfrac{\hat{k}}{k})^{\beta}(C_1 + \rho^{-1}\ln \hat{k})] \tag{30}$$

To summarize, we establish the following theorem,

THEOREM 1. *Based on Lemmas* 1*,* 2*, and* 3*, and suppose the goal of the government is to maximize the welfare of the representative agent, we have,* $\hat{\tau} = \rho^{-1}\ln[\beta(\tfrac{\hat{k}}{k})^{\beta}(C_1 + \rho^{-1}\ln \hat{k})]$*, where* $k = k(0) > 0$*,* $\hat{k}$ *is given in* (22)*,* β *is a solution of* $h(\beta) = 0$ *in* (14')*, and* C_1 *is given by Lemmas* 1 *and* 3.

REMARK 3.4. It follows from Theorem 1 that we confirm that the minimum time needed to "economic maturity" is endogenously determined and explicitly derived. In particular, the endogenous time depends on the following relevant parameters: the subject discount factor, the initial level of capital stock per capita, the utility-optimal and sustainable terminal path level of capital stock per capita, the natural growth rate of population, the exogenous level of government spending and also the volatility of the macro-economy. And one may, if motivated, develop more thorough comparative statics analyses of the endogenous time with respect to the above relevant parameters.
Noting that Theorem 1 is a conclusion of Step 4 of the computation algorithm in section 2.2, we now consider the following case corresponding to Step 3b of the computation algorithm.

CASE 2. *The goal of the government is to minimize the optimal stopping time of the representative agent*.
Thus, the problem facing the government can be expressed as,
PROBLEM 1. *The government is motivated to choose taxation policies so as to minimize the stopping time defined in* (30).
REMARK 3.5. Problem 1 is actually a nonlinear optimization problem and here we don't try to solve it due to its complication. It is easy to notice that different goals of the government usually lead to different fiscal policies, thereby resulting in different short-run economic consequences and even different speeds and paths of long-run economic development. And it is especially worth noting that there is a conjecture or possibility that the minimum time needed to "economic maturity", when the goal of the government is to minimize the endogenous time, may be much *longer* than that when the goal of the government is to maximize the welfare of the representative agent. Here we provide one reasonable explanation that the incentive or motive of investment of the representative agent may be terribly distorted when the goal of the government is not to maximize the welfare of the representative agent but to directly minimize the time needed to "economic maturity", thereby implying the micro-foundation of economic development is also distorted and hence retarding the speed of economic development. Therefore, the lesson for us is that for the government of an underdeveloped economy, choosing an appropriate development strategy and hence appropriate fiscal policies are of crucial importance in affecting and even determining the long-term speed and path of the *convergence* of the corresponding economical

system, and accordingly the long-term equilibrium level of the economy and welfare level of the representative agent.

4. Concluding remarks

The major goal of the current paper is to establish the minimum time needed to reach "economic maturity" for an underdeveloped economy in the background of stochastic endogenous growth. And the major novelties can be summarized as follows: first, the minimum time needed to "economic maturity" and the sustainable and utility-optimal terminal path level of capital stock per capita are simultaneously and endogenously determined; second, the endogenous time can be *explicitly computed* in some conditions, specifically for the preference of *the modified Radner fashion*, which will completely support comparative statics analyses; finally, it would be easily seen that the methodology introduced here can be easily employed to compute the optimal stopping times widely studied in finance.

What is more, we intuitively introduce some possible policy implications of the present results. It is plausible to argue that in an underdeveloped economy such as China (see, Song et al., 2011), the government and the households are motivated to choose appropriate fiscal policies and investment strategies, respectively, such that the economy reaches its maturity state as soon as possible. It is particularly worth emphasizing that the equilibrium minimum-time needed to economic maturity derived in the above theorem strictly depend on the initial value of the underlying economic system. This has to some extent reflected the well-known path-dependence effect analyzed and emphasized by North (1990). In other words, we argue that, besides in the process of institutional changes, path-dependence effect also plays a crucial role in economic development for those underdeveloped economies and Dai (2011) has analyzed the explicit dependence of the consumption process on the initial value of capital stock. In addition to that, as you can see in the above mathematical model, one can even proceed to the comparative static analysis of the equilibrium minimum-time needed to economic maturity with respect to the initial capital stock of the abstract economy. This of course will show us very interesting economic intuition and economic implication of the mathematical model. And it, therefore, would be regarded as an advantage of the framework established in the paper. Last but not least, the endogenous time depends on the following relevant parameters: the subject discount factor, the initial level of capital stock per capita, the utility-optimal and sustainable terminal path level of capital stock per capita, the natural growth rate of population, the exogenous level of government spending and also the volatility of the macro-economy.

Accordingly, the corresponding policy implications of the minimum-time needed to economic maturity can be explicitly computed and we have indeed established the micro-foundations of the economic development for those underdeveloped economies, which may throw some new insights into the understanding of the policy choice in reality. And it will be regarded as the major contribution of the present exploration.

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Appendix

A. Proof of Lemma 1

Applying the maximization operator in (11) yields,

$$1-\ g_p -\ r_s = \frac{1}{\exp(\rho(s+t))V_k(t,k(t))k(t)\mu g_p{}^{\alpha/1-\alpha}(1+\tau_c)}, \tag{A.1}$$

Substituting (A.1) into (11) gives,

$$-V_t(t,k(t)) - \tfrac{1}{2}\sigma^2 k^2(t)V_{kk}(t,k(t)) - \int_{R_0}\left[V\left(t,k(t)-\tfrac{\gamma z}{1+\gamma z}k(t)\right)-V(t,k(t))+\tfrac{\gamma z}{1+\gamma z}k(t)V_k(t,k(t))\right]n(dz)$$
$$= -\exp(-\rho(s+t))\left[\ln\left(e^{\rho(s+t)}V_k(t,k(t))\mu(1+\ \tau_c)\right)+1\right]$$
$$+V_k(t,k(t))k(t)[\mu g_p{}^{\alpha/1-\alpha}(1-\ \tau_k)-n+\sigma^2+b], \tag{A.2}$$

Naturally, one can try,

$$V(t,k(t)) = \exp(-\rho(s+t))[C_1 + C_2 \ln k(t)], \tag{A.3}$$

for some constants C_1, C_2 to be determined. Inserting (A.3) into (A.2) produces,

$$\rho C_1 + \rho C_2 \ln k(t) + \tfrac{1}{2}\sigma^2 C_2 - C_2\int_{R_0}\left(\ln\tfrac{1}{1+\gamma z}+\tfrac{\gamma z}{1+\gamma z}\right)n(dz)$$
$$= -\ln C_2 + \ln k(t) - \ln[\mu(1+\tau_c)] + C_2[\mu g_p{}^{\alpha/1-\alpha}(1-\ \tau_k)-n+\sigma^2+b]-\ 1, \tag{A.4}$$

which implies that $C_2 = \rho^{-1}$. And combining (A.4) with $C_2 = \rho^{-1}$ leads to,

$$C_1 = \rho^{-1}\left\{\ln\tfrac{\rho}{\mu(1+\tau_c)} + \rho^{-1}[\mu g_p{}^{\alpha/1-\alpha}(1-\tau_k)-n+\sigma^2+b]-1-\tfrac{1}{2}\sigma^2\rho^{-1}+\rho^{-1}\int_{R_0}\left(\ln\tfrac{1}{1+\gamma z}+\tfrac{\gamma z}{1+\gamma z}\right)n(dz)\right\}$$

Thus, the desired results are easily confirmed. ■

B. Proof of Lemma 2

It follows from the "Integro-variational inequalities for optimal stopping" (see, Theorem 2.2, pp. 29) of Øksendal and Sulem (2005), we are to prove,
(i) We need to prove that $\phi \geq g$ on D, i.e.,

$$Ck^{\beta} \geq \ln(g_p{}^{\alpha/1-\alpha}k) \text{ for } 0<k<\hat{k} \tag{B.1}$$

Define $l(k) \equiv Ck^{\beta} - \ln(g_p{}^{\alpha/1-\alpha}k)$. By our chosen values of C and $\hat{k}$, we see that $l(\hat{k}) = l'(\hat{k}) = 0$. Moreover, noting that $l''(k) = C\beta(\beta-1)k^{\beta-2} + k^{-2}$. Thus, if we put $\beta > 1$, we get $l''(k) > 0$ for $0<k<\hat{k}$, and also we have $l(k) > 0$ for all $0<k<\hat{k}$. Notice by (14) that,

$$h(1) = -\rho + \mu g_p{}^{\alpha/1-\alpha}[(1-\ \tau_k) - (1+\tau_c)(1-\ g_p -\ \hat{r}_s)] - n + \sigma^2 + b < 0$$

$$\Leftrightarrow \mu g_p{}^{\alpha/1-\alpha}[(1-\ \tau_k) - (1+\tau_c)(1-\ g_p -\ \hat{r}_s)] < n + \rho - \sigma^2 - b, \tag{B.2}$$

Thus, (B.1) follows as long as (B.2) is fulfilled.
(ii) Outside D we have $\phi(s,k) = e^{-\rho s}\ln(g_p{}^{\alpha/1-\alpha}k)$ and by (16),

$$Ag(s,k) = e^{-\rho s}\left\{-\rho\ln(g_p{}^{\alpha/1-\alpha}k) + \mu g_p{}^{\alpha/1-\alpha}[(1-\ \tau_k) - (1+\tau_c)(1-\ g_p -\ \hat{r}_s)]\right.$$

$$\left. -n + \tfrac{1}{2}\sigma^2 + b + d\right\} \leq 0 \text{ for all } k \geq \hat{k} \Leftrightarrow \hat{k} \geq g_p{}^{-\alpha/1-\alpha}\exp\left\{\frac{\mu g_p{}^{\alpha/1-\alpha}[(1-\ \tau_k)-(1+\tau_c)(1-\ g_p -\ \hat{r}_s)] - n + \frac{1}{2}\sigma^2 + b + d}{\rho}\right\}.$$

which holds by (20).
(iii) To check if $\hat{\tau} < \infty$ almost surely. By (4), (8) and Itô formula, we obtain,

$$k(t) = k\exp\Big(\Big\{\mu g_p{}^{\alpha/1-\alpha}[(1-\ \tau_k) - (1+\tau_c)(1-\ g_p -\ \hat{r}_s)] - n + \tfrac{1}{2}\sigma^2$$

$$+\int_R \gamma z n(dz)\Big\}t - \sigma B(t) + \int_0^t\int_{R_0}\ln(\tfrac{1}{1+\gamma z})N(ds,dz)\Big) \tag{B.3}$$

We see that if,

$$\mu g_p{}^{\alpha/1-\alpha}[(1-\ \tau_k) - (1+\tau_c)(1-\ g_p -\ \hat{r}_s)] > n - \tfrac{1}{2}\sigma^2 - \int_R \gamma z n(dz) \tag{B.4}$$

$$\gamma z < 0 \text{ a.s.} -n \tag{B.5}$$

And,

$$\sigma < 0. \tag{B.6}$$

by the law of the iterated logarithm of Brownian motion, then we have $\lim_{t\to\infty} k(t) = \infty$ a.s. And particularly, $\hat{\tau} < \infty$ almost surely.
(iv) Noting from (22) that $\hat{k} < \infty$, thus $[0,\hat{k}]$ is compact set by Heine-Borel theorem. Accordingly, ϕ is bounded on $[0,\hat{k}]$ via applying the fact that $\phi \in C^2(R^2)$ and the well-known Weierstrass theorem. So, it suffices to check that, $\{e^{-\rho\tau}\ln[g_p{}^{\alpha/1-\alpha}k(\tau)]\}_{\tau\in\mathrm{T}}$ is uniformly integrable on $[\hat{k},\infty)$. And T denotes the set of admissible stopping time and the uniform topology is naturally induced by the norm, which is induced by inner product, of Hilbert space $L^2(\Omega, F, P)$. For this to hold, it suffices to show that there exists a constant $M < \infty$ such that

$$E\{e^{-2\rho\tau}[\ln(g_p{}^{\alpha/1-\alpha}k(\tau))]^2\} \leq M \text{ for all } \tau \in \mathrm{T} \text{ and } k(\tau) \geq \hat{k}. \tag{B.7}$$

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Since $0<\ln[g_p{}^{\alpha/1-\alpha}k(t)]< g_p{}^{\alpha/1-\alpha}k(t)$ on $[\hat{k},\infty)$. Hence, by (4) and (B.3), we have,

$$E\{e^{-2\rho\tau}[\ln(g_p{}^{\alpha/1-\alpha}k(\tau))]^2\}\le g_p{}^{2\alpha/1-\alpha}E[e^{-2\rho\tau}k(\tau)^2]$$

$$=g_p{}^{2\alpha/1-\alpha}k^2E\Big[\exp\Big(\Big\{2\int_R\gamma zn(dz)+2\mu g_p{}^{\alpha/1-\alpha}[(1-\ \tau_k)-(1+\ \tau_c)(1-\ g_p-\ \hat{r}_s)]$$

$$-2n+3\sigma^2-2\rho+2\int_R\ln(\tfrac{1}{1+\gamma z})n(dz)\Big\}\tau+2\int_0^\tau\int_{R_0}\ln(\tfrac{1}{1+\gamma z})N(ds,dz)\Big)\Big] \qquad \text{(B.8)}$$

$$=g_p{}^{2\alpha/1-\alpha}k^2E\Big[\exp\Big(\Big\{2\int_R\gamma zn(dz)+2\mu g_p{}^{\alpha/1-\alpha}[(1-\ \tau_k)-(1+\ \tau_c)(1-\ g_p-\ \hat{r}_s)]$$

$$-2n+3\sigma^2-2\rho+2\int_R\ln(\tfrac{1}{1+\gamma z})n(dz)\Big\}\tau+\int_0^\tau\int_{R_0}[(\tfrac{1}{1+\gamma z})^2-1-2\ln(\tfrac{1}{1+\gamma z})]n(dz)ds\Big)\Big] \qquad \text{(B.9)}$$

$$=g_p{}^{2\alpha/1-\alpha}k^2E\Big[\exp\Big(\Big\{2\mu g_p{}^{\alpha/1-\alpha}[(1-\ \tau_k)-(1+\ \tau_c)(1-\ g_p-\ \hat{r}_s)]-2n+3\sigma^2-2\rho$$

$$+\int_{R_0}[(\tfrac{1}{1+\gamma z})^2-1]n(dz)+2\int_R\gamma zn(dz)\Big\}\tau\Big)\Big].$$

We conclude that if,

$$2\mu g_p{}^{\alpha/1-\alpha}[(1-\ \tau_k)-(1+\ \tau_c)(1-\ g_p-\ \hat{r}_s)]$$

$$\le 2n+2\rho-3\sigma^2-\int_{R_0}[(\tfrac{1}{1+\gamma z})^2-1]n(dz)-2\int_R\gamma zn(dz), \qquad \text{(B.10)}$$

Then (B.7) holds and so does (iv). Specifically, from (B.8) to (B.9), we have used the following fact. For the following equation,

$$dX(t)=X(t^-)\int_{R_0}(e^{\psi(t,z)}-1)N(dt,dz),\ X(0)=1. \qquad \text{(B.11)}$$

which has the solution,

$$X(t)=\exp\Big\{\int_0^t\int_{R_0}\psi(s,z)N(ds,dz)-\int_0^t\int_{R_0}[e^{\psi(s,z)}-1-\psi(s,z)]\nu(dz)ds\Big\} \qquad \text{(B.12)}$$

Suppose $\int_0^t\int_{R_0}(e^{\psi(s,z)}-1)^2\nu(dz)ds<\infty$. Then by (B.11) we see that $E[X(t)]=1$ and hence by (B.12) we obtain,

$$E\Big[\exp\Big(\int_0^t\int_{R_0}\psi(s,z)N(ds,dz)\Big)\Big]=\exp\Big\{\int_0^t\int_{R_0}[e^{\psi(s,z)}-1-\psi(s,z)]\nu(dz)ds\Big\}$$

If we put $\psi(s,z)=\ln(\frac{1}{1+\gamma z})^2$, then (B.9) follows.

(v) We need to prove that,

$$E^k\Big[|\phi(k(\tau))|+\int_0^{\hat{\tau}}\Big(\big|\hat{A}\phi(k(t))\big|+|-\sigma k(t)\phi_k(k(t))|^2$$

$$+\int_{R_0}\big|\phi(k(t)-\tfrac{\gamma z}{1+\gamma z}k(t))-\phi(k(t))\big|^2\nu(dz)\Big)dt\Big]<\infty \quad \text{for } \forall\tau\in \mathrm{T} \qquad \text{(B.13)}$$

where $\phi(k(t))=Ck(t)^\beta$ with C given in (23) and β satisfying $h(\beta)=0$ in (14). Noting that,

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$$\hat{A}\phi(k(t)) = \left\{\mu g_p{}^{\alpha/1-\alpha}[(1-\ \tau_k)-(1+\ \tau_c)(1-\ g_p-\ \hat{r}_s)]-n+\sigma^2+b\right\}k(t)\tfrac{\partial\phi}{\partial k}$$
$$+\tfrac{1}{2}\sigma^2 k(t)^2\tfrac{\partial^2\phi}{\partial k^2}+\int_{R_0}[\phi(k(t)-\tfrac{\gamma z}{1+\gamma z}k(t))-\phi(k(t))+\tfrac{\gamma z}{1+\gamma z}k(t)\tfrac{\partial\phi}{\partial k}]\nu(dz)$$
$$=[h(\beta)+\rho]\phi(k(t))=\rho\phi(k(t)). \tag{B.14}$$
$$\left|-\sigma k(t)\phi_k(k(t))\right|^2=(C\sigma\beta)^2 k(t)^{2\beta}, \tag{B.15}$$

And

$$\int_{R_0}\left|\phi(k(t)-\tfrac{\gamma z}{1+\gamma z}k(t))-\phi(k(t))\right|^2\nu(dz)=\int_{R_0}[(\tfrac{1}{1+\gamma z})^\beta-1]^2\nu(dz)[\phi(k(t))]^2, \tag{B.16}$$

Consequently, given,

$$\int_{R_0}[(\tfrac{1}{1+\gamma z})^\beta-1]^2\nu(dz)<\infty, \tag{B.17}$$

and via applying (iii), (B.13) follows as long as we show that $E^k[k(t)^{2\beta}]<\infty$ almost everywhere on $[0,\hat{\tau}]$. In particular, here we have $\beta>1$ by (B.2). Obviously, our following proof is similar to that of (iv). By (4) and (B.3), we have,

$$E^k[k(t)^{2\beta}]=k^{2\beta}E^k\Big[\exp\Big(\Big\{2\beta\int_R\gamma zn(dz)+2\beta\mu g_p{}^{\alpha/1-\alpha}[(1-\ \tau_k)-(1+\ \tau_c)(1-\ g_p-\ \hat{r}_s)]$$
$$-2\beta n+(\beta+2\beta^2)\sigma^2\Big\}t+2\beta\int_0^t\int_{R_0}\ln(\tfrac{1}{1+\gamma z})N(ds,dz)\Big)\Big]$$
$$=k^{2\beta}E^k\Big[\exp\Big(\Big\{2\beta\mu g_p{}^{\alpha/1-\alpha}[(1-\ \tau_k)-(1+\ \tau_c)(1-\ g_p-\ \hat{r}_s)]-2\beta n+(\beta+2\beta^2)\sigma^2$$
$$+\int_{R_0}[(\tfrac{1}{1+\gamma z})^{2\beta}-1]n(dz)+2\beta\int_R\gamma zn(dz)\Big\}t\Big)\Big].$$

Consequently, we show that if,

$$\Big|2\beta\mu g_p{}^{\alpha/1-\alpha}[(1-\ \tau_k)-(1+\ \tau_c)(1-\ g_p-\ \hat{r}_s)]+2\beta\int_R\gamma z\nu(dz)$$
$$-2\beta n+(\beta+2\beta^2)\sigma^2+\int_{R_0}[(\tfrac{1}{1+\gamma z})^{2\beta}-1]\nu(dz)\Big|<\infty, \tag{B.18}$$

Then we get $E^k[k(t)^{2\beta}]<\infty$ almost surely. ■